Spacecraft Attitude Determination Using a Decoupling Filter

Yong Joong Yoon*, Jae Weon Choi, Jang Gyu Lee*** and Tae Hyun Fang******

(*Received January 8, 1999*)

In this paper, an algorithm for real-time attitude estimation of spacecraft motion is investigated. For efficient computation, the decoupling filter presented in this paper is accomplished by a derived pseudo-measurement from the given measurement and the decoupled state in the original system. However, the proposed decoupling filter contains model errors due to coupling terms in the system. Therefore, we develope an attitude determination algorithm in which coupling terms are compensated through an error analysis. The attitude estimation algorithm using the state decoupling technique for real-time processing provides accurate attitude determination capability under a highly maneuvering dynamic environment, because the algorithm does not have any bias errors from a truncation, and the covariance of the estimator is compensated by nonlinear terms in the system. To verify the performance of the proposed algorithm vis-a-vis the EKF (extended Kalman filter), and the nonlinear filter, simulations have been performed by varying the initial values of the state and covariance, and measurement covariance. Results show that the proposed algorithm has consistently better performance than the EKF in all of the ranges of initial state values and covariance values of measurement, and it is as accurate as the nonlinear filter. However, the convergence speed of the nonlinear filter is faster than the proposed algorithm because of the pseudo-measurement model errors in the proposed algorithm. We show that the computational time of the proposed algorithm is improved by about 23% over the nonlinear filter.

Key Words: Kalman filter, Decoupling filter, Attitude estimation, Spacecraft

1. Introduction

The high maneuverability requirements of three-axis slewing spacecraft, when coupled with stringent attitude and pointing accuracy requirements, demand new techniques for the problem of spacecraft attitude determination. Another point of consideration is that it should operate in a real time on-board environment with only minimum ground interface in the nominal operation mode. This requirement imposes restrictions on the computational procedures for the real-time data reduction and processing. Based upon the above requirements, a real time on-board precision attitude algorithm has been developed to provide accurate attitude determination capability under a highly maneuvering dynamic environment (Lefferts, *et al*., 1982; Zwartbol, *et al*., 1985; Vathsal, 1987). However, useful algorithms in any practical application have not been realized. This paper derives an attitude estimator from the Fokker-Planck equation and reformulation of measurement.

The model of spacecraft attitude represented with quaternions is a second-order nonlinear system (Bar-Itzhack and Oshman, 1985; Vathal, 1987). The nonlinear filter for spacecraft attitude

* Managing Director, Planning & Operation Team, Satellite Busyress Center, Korea Telecom.

** Assistant Professor, School of Mechanical Engineering and Research Institute of Mechanical Technology, Pusan National University, Pusan 609-735, Korea.

*** Professor, Director, School of Electrical Engineering and Automatic Control Research Center, Seoul National University, Seoul 151-742, Korea.

**** Graduate Student, School of Mechanical Engineering and Research Institute of Mechanical Technology, Pusan National University, Pusan 609-735, Korea.

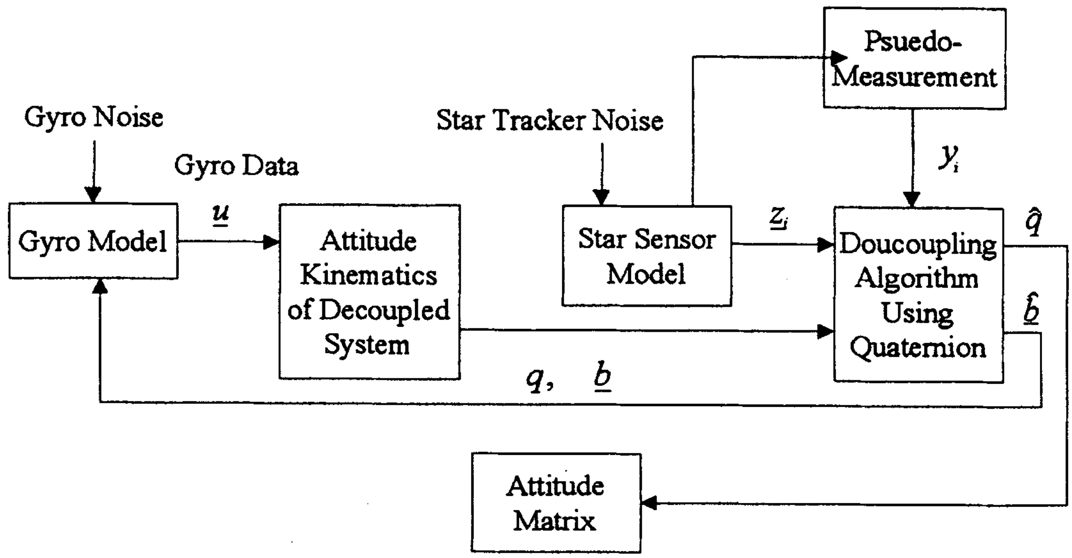

Fig. 1 Block diagram for attitude estimation using decopuling filter.

determination presented by Yoon, *et al.* (1998) does not produce any truncation bias errors, and the covariance of the estimator is compensated by the nonlinear terms of the system, since the propagation equation of attitude represented with quaternions is exactly derived by the Fokker-Planck equation (Jazwinski, 1970; Sage and Melsa, 1971). The nonlinear filter however requires a lot of computation because of an inherent nonlinearity and complexity of the system model for attitude. This paper introduces a new attitude estimation algorithm using the state decoupling technique for efficient computation. If the state variables in the attitude equation represented with quaternions can be decoupled into quaternion and gyro drift, then the nonlinear system of seven dimensions for attitude estimation can be transformed to two decoupled linear systems of dimensions four and three. The filter for the decoupled system requires two measurement equations related to quaternion and gyro drift. Accordingly, the decoupling filter needs the pseudo-measurement for gyro drift because the measurement equation using star sensors is only a function of quaternions. In this paper, pseudo-measurements for gyro drift are derived from the given measurement equation using the characteristics of the attitude matrix. The decoupling filter has large estimation errors by excluding the coupling terms of the state variables. Therefore, coupling terms are compensated through an error analysis. The decoupled system concept for attitude estimation is illustrated in the block diagram shown in Fig. 1.

In Sec. 2 and Sec. 3, we review the special features of the propagation and update equations of the nonlinear filter for attitude estimation. In Sec. 4, the equations of the decoupled filter are derived. An error analysis of the decoupling filter will be discussed in Sec. 5. In Sec. 6, we will discuss the simulation results and computational load for the proposed algorithm. Finally, in Sec. 7, the main resulfs will be summarized.

2. Propagation of Nonlinear Filter for Attitude Estimation

We assume that gyro errors are compensated by gyro calibration, so that the remaining gyro errors are only bias and white noise. Therefore, we will use the fact that the model of the gyro in spacecraft angular motions is related to the gyro output vector $\underline{u}$ according to the following relation (Heller, 1975; Lefferts and Markley, 1982):

$$\underline{u}(t)=\underline{\omega}(t)+\underline{b}(t)+\underline{\eta}_1(t) \tag{1}$$

The vector $\underline{\omega}$ is the true angular velocity, $\underline{b}$ is the drift rate bias, and $\underline{\eta}_1$ is the drift rate noise. The drift rate noise $\underline{\eta}_1$ is assumed to be a Gaussian white noise process. The drift rate bias is itself not a static quantity, but is driven by a second -Gaussian white noise process of the following form

$$\dot{\underline{b}}(t)=\underline{\eta}_2(t). \tag{2}$$

If the two noise processes are assumed to be uncorrelated, then we have

$$E[\underline{\eta}_1(t)\underline{\eta}_2(t)^T]=0. \tag{3}$$

Attitude determination for a spacecraft involves the estimation of the orientation of the spacecraft axes in space (Britting, 1971; Lefferts and Markley, 1982). This is achieved by processing the sensor's data on-board or on the ground. Commonly used attitude estimation methods for spacecraft are the Euler method, the direction cosine method, and the quaternion method (Nurse, *et al.*, 1978). Among them, the quaternion method is the most popular because of its advantages with respect to nonsingularity, simplicity, and computation time (Miller, 1978; Nurse, *et al.*, 1978). In the system investigated, the attitude is represented by a quaternion defined as

$$q=\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}=\begin{bmatrix} \phi_x/\phi_0 \sin(\phi_0/2) \\ \phi_y/\phi_0 \sin(\phi_0/2) \\ \phi_z/\phi_0 \sin(\phi_0/2) \\ \cos(\phi_0/2) \end{bmatrix}, \tag{4}$$

where ϕ_x, ϕ_y, and ϕ_z are elements of the vector $\underline{\phi}$ which is a rotational unit vector related to the rotation axes, and the angle ϕ_0 is the magnitude of the rotational vector. The quaternion possesses three degrees of freedom and satisfies the following constraint:

$$q^Tq=1. \tag{5}$$

The quaternion differential equation is given by (Miller, 1978)

$$\dot{q}=1/2\Omega(\underline{\omega})q, \tag{6a}$$

where $\Omega(\underline{\omega})$ is a skew symmetric matrix given by

$$\Omega(\underline{\omega})=\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}. \tag{6b}$$

If we assume that the spacecraft attitude states are given by the attitude quaternion and the gyro drift rate bias vector, then the dimension of the attitude system is seven:

$$\underline{x}(t)\equiv\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}=\begin{bmatrix} q(t) \\ \underline{b}(t) \end{bmatrix}. \tag{7}$$

The quaternion and the bias vector have been shown to satisfy the following coupled differential equations (Lefferts and Markley, 1982; Gai, *et al.*, 1985):

$$\dot{q}(t)=1/2\Omega(\underline{u}(t)-\underline{b}(t)-\underline{\eta}_1(t))q(t) \tag{8a}$$

$$\dot{\underline{b}}(t)=\underline{\eta}_2(t). \tag{8b}$$

Let us express errors of the state $\underline{x}$ as follows:

$$\begin{bmatrix} \Delta\underline{x}_1(t) \\ \Delta\underline{x}_2(t) \end{bmatrix}=\begin{bmatrix} q(t)-\hat{q}(t) \\ \underline{b}(t)-\hat{\underline{b}}(t) \end{bmatrix} \tag{9}$$

which is an implied definition of $\Delta\underline{x}$. Substitution of Eq. (9) into Eq. (8) yields

$$\begin{aligned}\begin{bmatrix} \dot{\underline{x}}_1(t) \\ \dot{\underline{x}}_2(t) \end{bmatrix}=&\begin{bmatrix} 1/2\Omega(\underline{u}-\hat{\underline{x}}_2(t)) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\underline{x}}_1(t) \\ \hat{\underline{x}}_2(t) \end{bmatrix} \\ &+\begin{bmatrix} 1/2\Omega(\underline{u}-\hat{\underline{x}}_2(t)) & 1/2\Gamma(\hat{\underline{x}}_1(t)) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta\underline{x}_1(t) \\ \Delta\underline{x}_2(t) \end{bmatrix} \\ &+\begin{bmatrix} -1/2\Gamma(\hat{\underline{x}}_1(t)) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \underline{\eta}_1(t) \\ \underline{\eta}_2(t) \end{bmatrix} \\ &+\begin{bmatrix} -1/2\Gamma(\Delta\underline{x}_1(t)) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \underline{\eta}_1(t) \\ \underline{\eta}_2(t) \end{bmatrix} \\ &+\begin{bmatrix} -1/2\Omega(\Delta\underline{x}_2(t))\Delta\underline{x}_1(t) \\ 0 \end{bmatrix},\end{aligned} \tag{10}$$

where

$$\Gamma(q)=\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}.$$

When we apply Eq. (10) to the propagation equation derived from the Fokker-Planck equation, the propagation equation of the state is exactly obtained as follows (Sage and Melsa, 1971; Vathsal, 1987):

$$\begin{aligned}\dot{\hat{\underline{x}}}(t)&=E[f(\underline{x})] \\ &=\begin{bmatrix} 1/2\Omega(\underline{u}-\hat{\underline{x}}_2(t)) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\underline{x}}_1(t) \\ \hat{\underline{x}}_2(t) \end{bmatrix} \\ &\quad+E\begin{bmatrix} -1/2\Omega(\Delta\underline{x}_2(t))\Delta\underline{x}_1(t) \\ 0 \end{bmatrix}\end{aligned} \tag{11a}$$

where, $f(\underline{x})$ is the right-half term of Eq. (10), and

$$E[-1/2\Omega(\Delta\underline{x}_2(t))\Delta\underline{x}_1(t)] = 1/2\begin{bmatrix} -p_{27}+p_{36}-p_{45} \\ p_{17}-p_{35}-p_{46} \\ -p_{16}+p_{25}-p_{47} \\ p_{15}+p_{26}+p_{37} \end{bmatrix} \quad (11b)$$

where p_{ij} is the element of the error covariance matrix of state estimates. The prediction values of the state vector obtained from Eq. (11) have the properties that they do not contain truncation errors due to nonlinearity of the system for attitude dynamics, and possess a simple form amenable to solving Eq. (11).

The differential equation for the seven-dimensional error covariance matrix P is exactly derived from the Fokker-Planck equation as follows (Sage and Melsa, 1971):

$$\dot{P}=E[f\Delta\underline{x}^T]+E[\Delta\underline{x}f^T]+E[GQG^T]. \quad (12)$$

where G is the process noise matrix and Q is the covariance matrix of the process noise.

The solution of Eq. (12) is employed by expanding the function $f(x, t)$ by a Taylor series (Jazwinski, 1970; Sage and Melsa, 1971), but is difficult and complex. If we let the state errors be Gaussian, then the continuous propagation equation of the covariance P can be obtained by the substitution of Eq. (10) into Eq. (12). However, it is difficult to propagate P in the seven-dimensional state space, since the system matrix of the attitude is a singular matrix. A transformation matrix was derived by Lefferts *et al.*(1982) that propagates the error covariance matrix in six -dimensional state space. From the result, we can obtain the covariance propagation equation of six dimensions P' as follows:

$$\dot{P}'=F'P'+P'(F')^T+G'Q(G')^T+M' \quad (13a)$$

where,

$$S(\hat{\underline{x}}_1)=\begin{bmatrix}\Gamma(\hat{\underline{x}}_1(t)) & 0 \\ 0 & I\end{bmatrix}_{7\times 6} \quad (13b)$$

$$F'(t)=\begin{bmatrix}[\hat{\omega}(t)x] & -1/2I \\ 0 & 0\end{bmatrix} \quad (13c)$$

$$G'(t)=\begin{bmatrix}-1/2I & 0 \\ 0 & I\end{bmatrix} \quad (13d)$$

$$[\underline{\omega}(t)x]=\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \quad (13e)$$

$$M'(t)=\begin{bmatrix} \alpha_y p'_{33}+\alpha_z p'_{22} & -\alpha_z p'_{12} & -\alpha_y p'_{13} \\ -\alpha_z p'_{12} & \alpha_x p'_{33}+\alpha_z p'_{11} & -\alpha_x p'_{23} \\ -\alpha_y p'_{13} & -\alpha_x p'_{23} & \alpha_y p'_{11}+\alpha_x p'_{22} \end{bmatrix} \quad (13f)$$

where α is the variance of gyro. $[\underline{\omega}(t)x]$ (Britting, 1971) is the skew symmetric matrix. The derived Eq. (13) is a recursive form of the six-dimensional differential covariance equation for a general gyro model.

3. Discrete Update Equation

We assume that the spacecraft is equipped with two precision star trackers for obtaining attitude information of the spacecraft in inertial space. The measured value of a star sensor in the body frame can be written as the following equation (Lefferts, *et al.*, 1982; LO, 1986; Vathsal, 1987):

$$\underline{z}=TA_{BS}(q_c)A_{SI}(q)\underline{\rho}+\underline{v}', \quad (14)$$

where $A_{SI}(q)$ is the spacecraft attitude matrix, $A_{BS}(q_c)$ is the alignment matrix of the star tracker, and $\underline{\rho}$ is a star position vector in the reference coordinate system. $\underline{v}'$ is a random noise vector due to star catalog position errors and star tracker output noises. The attitude matrix with the quaternion in Eq. (14) is a second-order nonlinear function. We will attempt a reformulation of the measurement equation in which measurement noise implies the nonlinearity of the mesurement. Accordingly, if we replace the states $\underline{x}$ in Eq. (14) with $\hat{\underline{x}}+\Delta\underline{x}$, then the measurement equation is rewritten as follows:

$$\underline{z}_i=TA_{BS}(q_c)\{A_{SI}(\hat{\underline{x}}_1)\,{}_i\underline{\rho}_i+H_i(\hat{\underline{x}}_1, \underline{\rho}_i)\Delta\underline{x}_1\}+\underline{v}_i, \quad (15a)$$

where

$$T=\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix} \quad (15b)$$

$$H_i(\hat{\underline{x}}_1, \underline{\rho}_i)=\frac{\partial A_{SI}(\underline{x}_1)\underline{\rho}_i}{\partial \underline{x}_1}\Big|_{\underline{x}_1=\hat{\underline{x}}_1} \quad (15c)$$

$$\underline{v}=TA_{BS}(q_c)A_{SI}(\Delta\underline{x}_1)\rho_i+\underline{v}'. \quad (15d)$$

The redefined measurement noise of Eq. (15c) is not a white Gaussian noise, but a colored noise. In Eq. (15c), let elements of the star position vector be ρ_x, ρ_y, and ρ_z, respectively; then we can obtain the stochastic mean value of the colored measurement. That is,

$$E[\underline{v}] = TA_{BS}(q_c)\,E[A_{SI}(\Delta\underline{x}_1)\,\underline{\rho}_i]$$
$$= TA_{BS}(q_c) \quad (16)$$
$$\times \begin{bmatrix} (p_{11}-p_{22}-p_{33}+p_{44})\rho_x+2(p_{12}+p_{34})\rho_y+2(p_{13}-p_{24})\rho_z \\ 2(p_{12}-p_{34})\rho_x+(-p_{11}+p_{22}-p_{33}+p_{44})\rho_y+2(p_{14}+p_{23})\rho_z \\ 2(p_{13}+p_{24})\rho_x+2(p_{23}-p_{14})\rho_y+(-p_{11}-p_{22}+p_{33}+p_{44})\rho_z \end{bmatrix}$$

where the element p_{ij} are the covariance elements corresponding to state i and j. We assume that state error $\Delta\underline{x}$ is Gaussian; then the covariance R of the colored measurement noise can be obtained as follows:

$$R = TA_{BS}(q_c)\{L(P, \underline{\rho})\}A_{BS}(q_c)^T T^T + R'. \quad (17)$$

The elements of the L matrix in Eq. (17) consist of a star position vector and covariance elements. Note that the derived measurement equation (Eq. (15a)) can be very easily determined from the star position vector, and recursive formulas are readily applicable. By observing the measurement equation in Eq. (15a), one can see that our intension is to correct the model error due to the nonlinear terms of the measurement.

The discrete Kalman update equation of the states is given by

$$\underline{\hat{x}}_k(+) = \underline{\hat{x}}_k(-) + K_k\{\underline{z}_k - \underline{\hat{z}}_k\}. \quad (18)$$

Substituting Eq. (15a) into Eq. (18), the state update equation is rewritten as

$$\underline{\hat{x}}_k(+) = \underline{\hat{x}}_k(-) + K_k\{\underline{z}_k - TA_{BS}(q_c)\,A_{SI}(\underline{\hat{x}}_1)\,{}_i\underline{\rho}_i - E[\underline{v}_i]\}$$
$$= \underline{\hat{x}}_k(-) + K_k\{\underline{z}_k - h_k(\underline{\hat{x}}_1) - \pi_k\}, \quad (19)$$

where $h_k(\underline{\hat{x}}_1) = TA_{BS}(q_c)\,A_{SI}(\underline{\hat{x}}_1)\,{}_i\underline{\rho}_i$ and $\pi_k = E[\underline{v}_i]$. We already know from Eq. (11) and Eq. (15a), that $x_k(-)$ in Eq. (19) has no truncation errors, and the residual of the measurement also does not have truncation errors. Therefore, the derived state update equation (Eq. (19)) dose not have truncation errors due to the nonlinear terms of the system and measurement. Let us define the update covariance P as (Jazwinski, 1970; Sage and Melsa, 1971; Lewis, 1986)

$$P_k(+) = E[\Delta\underline{x}_k(+)\,\Delta\underline{x}_k^T(+)], \quad (20a)$$

where

$$\Delta\underline{x}_k(+) = \underline{x}_k - \underline{\hat{x}}_k(+)$$
$$= \Delta\underline{x}_k(-) - K_k\{TA_{BS}(q_c)\,H_k \times (\underline{\hat{x}}_k, \rho)\,\Delta\underline{x}_k + \underline{v}_k - \pi_k\}. \quad (20b)$$

If we neglect the third moment of $\Delta\underline{x}_k$ under a Gaussian assumption, then we can obtain the update covariance equation as follows:

$$P_k(+) = P_k(-) - P_k(-)\,H_k K_k^T - K_k H_k P_k(-) + K_k\{H_k P_k(-)\,H_k^T + R_k\}K_k^T, \quad (21)$$

where R_k is a covariance matrix with a colored noise due to the nonlinearity of the measurement. Therefore, we can obtain the following optimal gain under a Gaussian assumption:

$$K_k = P_k(-)\,H_k^T\{H_k P_k(-)\,H_k^T + R_k\}^{-1}. \quad (22)$$

Substituting Eq. (22) into Eq. (21), we can obtain the update covariance equation as follows:

$$P_k(+) = \{I - K_k H_k\}P_k(-). \quad (23)$$

We see that Eq. (23) is the same form as a linear system, and it is compensated by the time-varying nonlinearity of the system and measurement. However, the update covariance equation neglected the third moment, the proposed nonlinear filter is accordingly a suboptimal filter because it does not contain truncation errors due to the nonlinearity.

4. Decoupling Filter Equation

The nonlinear filter in Sec. 2 and 3 requires a signiticant amonst of computation because of the inherent nonlinearity and complexity of the system model for attitude estimation. For efficient computation, if we assume that the states in Eq. (8) are decoupled by quaternion and gyro drift terms, then the model of the decoupled system is given by

$$\dot{q} = 1/2\Omega(\underline{u} - \underline{b}^m)\,q^m + 1/2\Omega(\underline{u} - \underline{b}^m)\,\Delta q^m - 1/2\Gamma(q^m)\,\underline{\eta}_1 - 1/2\Gamma(\Delta q^m)\,\underline{\eta}_1 \quad (24a)$$
$$\dot{b} = \underline{\eta}_2. \quad (24b)$$

Also, the decoupled covariance equation corresponding to Eq. (12) is obtained as follows:

$$\dot{P}_q^n = 1/2\Omega(\underline{u}-\underline{b}^m)P_q^m + 1/2P_q^m\Omega^T(\underline{u}-\underline{b}^m) + 1/4\Gamma(q^m)Q_1\Gamma^T(q^m) + 1/4E[\Gamma(q^m)Q_1\Gamma^T(q^m)] \quad (25a)$$

$$\dot{P}_b^m = Q_2. \quad (25b)$$

The decoupled filter requires the pseudo-measurement for gyro drift and the measurement equation of a star sensor. To derive the pseudo-measurement for gyro drift from Eq. (15a), we will use the following relation between the attitude matrix and angular velocity (Britting, 1971):

$$\dot{A}(q) = -[\underline{\omega}(t)x]A(q). \quad (26)$$

The discretized form of Eq. (26) is given by

$$A_i(q) = TA_{BS}\{I - \Delta t[\underline{\omega}x]_{i-1}\}_{SI}A(q)_{i-1}. \quad (27)$$

Substituting Eq. (27) into Eq. (15a), the discrete measurement equation is rewritten as follows:

$$\underline{z}_i = TA_{BS}(q_c)\{I - \Delta t[\underline{\omega}x]_{i-1}\}A(q)_{i-1}\underline{\rho}_i + \underline{v}_i'. \quad (28)$$

The right side of Eq. (28) is multipied by the identity matrix $A_{BS}A_{BS}^T$, and applying the relation between the measurement value and true value of the star sensor to Eq. (29), the measurement equation is given by Eq. (30) (Gai, *et al*., 1985):

$$\underline{z}_{i-1} = TA_{BS}(q_c)A_{SI}(q)_{i-1}\underline{\rho}_{i-1} + \underline{v}_{i-1}' = T\underline{z}_{i-1}^* + \underline{v}_{i-1}' \quad (29)$$

$$\underline{z}_i = TA_{BS}(q_c)\{I - \Delta t[\underline{\omega}x]_{i-1}] \times A_{BS}^T(q_c)\underline{z}_{i-1}^* + \underline{v}_i'. \quad (30)$$

Inspecting Eq. (30), we can find that the measurement equation does not include the position vector of the tracked star. Therefore the derived measurement equation does not require identification algorithm with much computation time for the tracked star. Substituting Eq. (1) into Eq. (30), the measurement related to gyro drift is given by Eq. (31):

$$\underline{z}_i = TA_{BS}(q_c)\{I - \Delta t[(\underline{u}_i - \underline{b}_{i-1} - \underline{\eta}_{1,i-1})x]\} A_{BS}^T(q_c)\underline{z}_{i-1}^* + \underline{v}_i'. \quad (31)$$

Let us simplify Eq. (31) through an auxiliary variable defined by Eq. (32); then the pseudo-measurement for gyro drift can be rewritten as Eq. (33):

$$\underline{y}_i = \underline{z}_i - TA_{BS}(q_c)[I - \Delta t\underline{u}_{i-1}x]A_{BS}^T(q_c)\underline{z}_{i-1}^* - \underline{v}_{i-1}' \quad (32a)$$

$$= \underline{z}_i - \underline{z}_{i-1} + TA_{BS}(q_c)\Delta t[\underline{u}_{i-1}x]A_{BS}^T(q_c)\underline{z}_{i-1}^*$$

$$\underline{\varepsilon}_i = \Delta tTA_{BS}(q_c)[\underline{\eta}_{1,i-1}x]A_{BS}^T(q_c)\underline{z}_{i-1}^* + \underline{v}_i' - \underline{v}_{i-1}' \quad (32b)$$

$$\underline{y}_i = TA_{BS}(q_c)\Delta t[\underline{b}_{i-1}x]A_{BS}^T(q_c)\underline{z}_{i-1}^* + \underline{\varepsilon}_i \quad (33)$$

where the mean and covariance of the pseudo-measurement noise $\underline{\varepsilon}_i$ are calculated by

$$E[\underline{\varepsilon}_i] = E[\Delta tTA_{BS}(q_c)[\underline{\eta}_{1,i-1}x] \times A_{BS}^T(q_c)\underline{z}_{i-1}^* + \underline{v}_i - \underline{v}_{i-1}] = 0 \quad (34a)$$

and

$$E[\underline{\varepsilon}_i\underline{\varepsilon}_i^T] = TA_{BS}(q_c)E[\Delta t^2[\underline{\eta}_1x]_{i-1}A_{BS}^T(q_c)\underline{z}_{i-1}^*(\underline{z}_{i-1}^*)^T \times A_{BS}(q_c)[\underline{\eta}_1x]_i^TA_{BS}^T(q_c)T^T + 2R_i']. \quad (34b)$$

If we assume that the alignment matrix, $A_{BS}(q_c)$, is an identity matrix, then the pseudo-measurement can be represented as follows:

$$\underline{y}_i = \Delta t\begin{bmatrix} 0 & -\underline{z}_z^* & \underline{z}_y^* \\ \underline{z}_z^* & 0 & -\underline{z}_x^* \end{bmatrix}_{i-1}\underline{b}_i + \underline{\varepsilon}_i = H_b b_i + \underline{\varepsilon}_i, \quad (35a)$$

where

$$\underline{y}_i = \underline{z}_i - \underline{z}_{i-1} + \Delta tT[\underline{u}_{i-1}x]\underline{z}_{i-1}^* \quad (35b)$$

$$E[\underline{\varepsilon}_i\underline{\varepsilon}_i^T] = \Delta t^2\begin{bmatrix} a_3 + a_2 & 0 \\ 0 & a_1 + a_3 \end{bmatrix} + 2R_i'. \quad (35c)$$

Since $\underline{z}^*$ in Eq. (35) is not available because of the measurement noise, $\underline{z}^*$ is replaced by $\underline{z}$ in the pseudo-measurement formulation. If $\underline{z}$ is used in Eq. (35) rather than $\underline{z}^*$, then the errors in Eq. (35) reduce to those of Eq. (36):

$$\Delta t[\underline{u}x]\underline{v}', \; \Delta t[\underline{b}x]\underline{v}' \quad (36)$$

where the magnitudes of Δt, $\underline{u}$, $\underline{b}$, and $\underline{v}'$ generally are smaller than unity. Therefore, we can neglect the error in Eq. (35) because the relation of the pseudo-measurement errors are given by

$$\Delta t[\underline{u}x]\underline{v}' \ll \underline{\varepsilon}, \; \Delta t[\underline{b}x]\underline{v}' \ll \underline{\varepsilon}. \quad (37)$$

The gyro drift filter can be obtained by the gyro drift equation in Eq. (24) and the pseudo-measurement equation in Eq. (35), and the quaternion filter can be obtained by the quaternion equation in Eq. (24a) and the measurement equation of the star sensor in Eq. (15a). Therefore, the decoupling filter equations are represented by the Table 1 (Jazwinski, 1970; Sage and Melsa, 1971; Lewis, 1986).

Table 1 Decoupling filter equations for attitude estimation.

State	Propagation equation	Update equation
Gyro drift	$\dot{\underline{b}}^m=0$ $\dot{P}_b^m=Q$ Measurement:- Filter Gian: -	$\underline{b}^m(+)=\underline{b}^m(-)+K_b^m(\underline{y}_i-\hat{\underline{y}}_i)$ $P_b^m(+)=P_b^m(-)-K_b^m H_b P_b^m(-)$ $\underline{y}_i=H_b\underline{b}_i+\underline{\varepsilon}_i$ $K_b^m=P_b^m(-)H_b^T[H_b P_b^m(-)H_b^T+R\varepsilon]^{-1}$
Quater-nion	$\dot{q}^m=1/2(\underline{u}-\underline{b}^m)q^m$ $\dot{P}_q^m=F_q P_q^m+P_q^m F^T$ $+GQ_1G^T+M$ Measurement: - Filter Gain: -	$q^m(+)=q^m(-)+K_q^m(\underline{z}_i-\hat{\underline{z}}_i)$ $P_q^m(+)=P_q^m(-)-K_q^m H_q P_q^m(-)$ $\underline{z}_i=TA_{BS}(q_c)A_{SI}(q)\underline{\rho}_i+\underline{v}_i'$ $K_q^m=P_q^m(-)H_q^T[H_q P_q^m(-)H_q^T+R]^{-1}$

5. Algorithm and the Error Analysis of Decoupling Filter

It is clear that an inexact decoupling filter model that neglects coupling terms will degrade filter performance. In fact, such an inexact model may cause the filter to diverge. It is therefore important to evaluate the effect of coupling terms on performance of the designed decoupling filter (Jazwinski, 1970; Lewis, 1986).

In this section, we will minimize model error through error analysis of the decoupling filter. Suppose that the estimation errors of the decoupling filter are described by

$$q=q^m+\Delta q^m \tag{38a}$$

$$\underline{b}=\underline{b}^m+\Delta\underline{b}^m. \tag{38b}$$

Substituting Eq. (38) into Eq. (8), the model of the actual system is obtained as follows:

$$\begin{aligned}\dot{q}=&1/2\Omega(\underline{u}-\underline{b}^m)q^m+1/2\Omega(\underline{u}-\underline{b}^m)\Delta q^m\\&-1/2\Gamma(q^m)\Delta\underline{b}^m\\&-1/2\Gamma(\Delta\underline{b}^m)\Delta q^m-1/2\Gamma(q^m)\underline{\eta}_1\\&-1/2\Gamma(\Delta q^m)\underline{\eta}_1\end{aligned} \tag{39a}$$

$$\dot{\underline{b}}=\underline{\eta}_2. \tag{39b}$$

In addition, the decoupled system model was given by Eq. (24). From Eq. (24) and Eq. (39), we can see that the model of the decoupled system differs from the actual model. Now the computed covariance matrix P^m in the decoupling filter is not the estimation error covariance matrix, since the filter model differs from the actual model. Thus, the filter is not the minimum variance filter for the actual system of Eq. (39). A measure of the designed filter performance is provided by the actual estimation error covariance matrix as

$$\begin{aligned}\dot{P}_q=&1/2\Omega(\underline{u}-\underline{b}^m)P_q^m+1/2P_q^m\Omega^T(\underline{u}-\underline{b}^m)\\&-1/2\Gamma(q^m)P_{bq}^m\\&-1/2P_{bq}^m\Gamma^T(q^m)+1/4\Gamma(q^m)Q_1\Gamma^T(q^m)\\&+1/4E[\Gamma(q^m)Q_1\Gamma^T(q^m)]\end{aligned} \tag{40a}$$

$$\dot{P}_{bq}=1/2\Omega(\underline{u}-\underline{b}^m)P_{bq}^m-1/2\Gamma(q^m)P_b^m \tag{40b}$$

$$\dot{P}_b=Q_2. \tag{40c}$$

Inspecting Eq. (25) and Eq. (40), we can find that the actual error variance is less than or equal to the designed error variance since the decoupling filter does not contain the correlation matrix P_{bq} in Eq. (25). In this case, the stability of the designed filter can not be guaranteed.

To improve performance of the decoupling filter, we will compute the correlation matrix P_{bq} using the update equation of the decoupling filter, and the decoupling filter will be compensated by computing P_{bq}. The update equation of the filter for the actual system for attitude estimation is given by

$$K=\begin{bmatrix}K_q\\K_b\end{bmatrix}=\begin{bmatrix}P_q(-)H_q^T(H_qP_q(-)H_q^T+R)^{-1}\\P_{bq}(-)H_q^T(H_qP_q(-)H_q^T+R)^{-1}\end{bmatrix} \tag{41a}$$

$$\hat{q}(+)=\hat{q}(-)+\hat{K}_q(\underline{z}_i-\hat{\underline{z}}_i) \tag{41b}$$

$$\hat{\underline{b}}(+)=\hat{\underline{b}}(-)+K_b(\underline{z}_i-\hat{\underline{z}}_i) \tag{41c}$$

$$P_b(+)=P_q(-)-K_qH_qP_q(-) \tag{41d}$$

$$P_{bq}(+)=P_{bq}(-)-K_bH_qP_q(-) \tag{41e}$$

$$P_b(+)=P_b(-)-K_bH_qP_{bq}^T(-). \tag{41f}$$

Also, Table 1 gives the update equation of the decoupling filter for attitude estimation.

The predicted values of gyro drift have the same quantities since the model of gyro drift in the decoupling filter is the same as the model of the actual system, and we can assume that the filter gain K_b in Eq. (41) equals K_b^m since the derived pseudo-measurement in the Sec. 3 is nearly exact except for the effect of measurement noise. If the correlation term P_{bq} in Eq. (25) is compensated, then $P_q(-)$ in Eq. (41) equals $P_q^m(-)$ in Eq. (25), also K_q in Eq. (41a) equals $K_q^m(-)$ in the Table 1. Applying these results to Eq. (41a), Eq. (41b), and the decoupling filter gain in Table 1, the correlation matrix $P_{bq}(-)$ can be computed as

$$P_{bq}(-)=K_b^m(K_q^m)^T\{K_q^m(K_q^m)^T\}^{-1}P_q^m(-). \tag{42}$$

Similarly, we can obtain the update equation of the correlation matrix $P_{bq}(+)$ as

$$P_{bq}(+)=P_{bq}(-)-K_b^m H_q P_q^m(-). \tag{43}$$

If the decoupling filter can be compensated by the computed correlation matrix in Eq. (25a) and the update covariance equation in Table 1, then the proposed filter will minimize the model error of the decoupling filter. Therefore, the attitude algorithm using the compensated decoupling filter can be summarized as follows:

$$\dot{\hat{q}}^s=1/2\Omega(\underline{u}-\hat{\underline{b}}^s)\,\hat{q}^s-1/2E[\Omega(\Delta\underline{b}^s)\Delta q^s] \tag{44a}$$

$$\dot{\hat{\underline{b}}}^s=\underline{0} \tag{44b}$$

$$\begin{aligned}\dot{P}_q^s=&1/2\Omega(\underline{u}-\hat{\underline{b}}^s)P_q^s-1/2\Gamma(\hat{q}^s)P_{bq}^s\\&+1/2P_q^s\Omega^T(\underline{u}-\hat{\underline{b}}^s)-1/2P_{bq}^s\Gamma^T(\hat{q}^s)\\&+1/4\Gamma(\hat{q}^s)Q_1\Gamma^T(\hat{q}^s)+1/4E[\Gamma(\Delta q^s)\\&\quad Q_1\Gamma^T(\Delta q^s)]\end{aligned} \tag{44c}$$

$$\dot{P}_b^s=Q_2 \tag{44d}$$

$$\hat{\underline{q}}^s(+)=\hat{\underline{q}}^s(-)+K_q^s(\underline{z}_i-\hat{\underline{z}}_i) \tag{44e}$$

$$\hat{\underline{b}}^s(+)=\hat{\underline{b}}^s(-)+K_b^s(\underline{y}_i-\hat{\underline{y}}_i) \tag{44f}$$

$$P_q^s(+)=P_q^s(-)-K_q^sH_qP_q^s(-) \tag{44g}$$

$$P_{bq}^s(-)=K_b^s(K_b^s)^T\{K_q^s(K_q^s)^T\}^{-1}P_q^s(-) \tag{44h}$$

$$P_{bq}^s(+)=P_{bq}^s(-)-K_b^sH_qP_q^s(-) \tag{44i}$$

$$P_b^s(+)=P_b^s(-)-K_b^sH_bP_b^s(-). \tag{44j}$$

But the proposed algorithm in Eq. (44) has an estimation error due to the model error of the pseudo-measurement, and uses constant values of P_{bq} between the cross-measurement interval. Therefore, the proposed algorithm using the decoupling filter cannot be an optimal estimator.

6. Simulation Results

Performance of the proposed attitude algorithm in Sec. 5 is verified through the flow diagram shown in Fig. 2. We assume that the orbital period of a spacecraft is 120 min and the spacecraft has two star trackers, with the angular velocity of the spacecraft sensed by the gyros 0.05 deg/sec. Hence $\omega_1=0$, $\omega_2=0.005$, and $\omega_3=0$ deg/sec are used for the simulation. Attitude sensors, when operating under high and continuous slew rates, acceleration, and jerk motions, may introduce significant errors because of the coupled terms of three axes of a spacecraft (Lefferts, *et al*., 1982; Zwartbol, *et al*., 1985; Vathsal, 1987). Therefore, to verify the performance of the proposed algorithm and EKF, simulations have been performed by varying the initial state values of the filters given in Table 2; the simulation results are depicted in the corresponding figures. Also, the covariance initial values corresponding to the state initial values are given by

$$\delta\phi_0=5^\circ;\; P_0=\begin{bmatrix}6.4\times10^{-4}I_{4\times4} & 0\\ 0 & 5.3\times10^{-11}I_{3\times3}\end{bmatrix}$$

$$\delta\phi_0=10^\circ;\; P_0=\begin{bmatrix}2.6\times10^{-3}I_{4\times4} & 0\\ 0 & 5.3\times10^{-11}I_{3\times3}\end{bmatrix}$$

$$\delta\phi_0=15^\circ;\; P_0=\begin{bmatrix}5.7\times10^{-3}I_{4\times4} & 0\\ 0 & 5.3\times10^{-11}I_{3\times3}\end{bmatrix}.$$

The gyro noise and the measurement noise have been simulated using RAND and GAUSS subroutines that generate uniformly distributed random numbers and Gaussian-distributed random numbers, respectively. The gyro data has been simulated for a sampling time of 500 msec and star outputs are simulated at an interval of 120 sec. The covariance propagation equations have been simulated with a step size of 500 msec using a fourth-order Runge-Kutta scheme of numerical integration on a digital computer. The standard deviation of the process noise $\underline{\eta}_1$ has been simulated for 1 arc second/sec. The standard deviation of the measurement noise R' has been assumed to lie between 10 arc seconds to 200 arc seconds. The

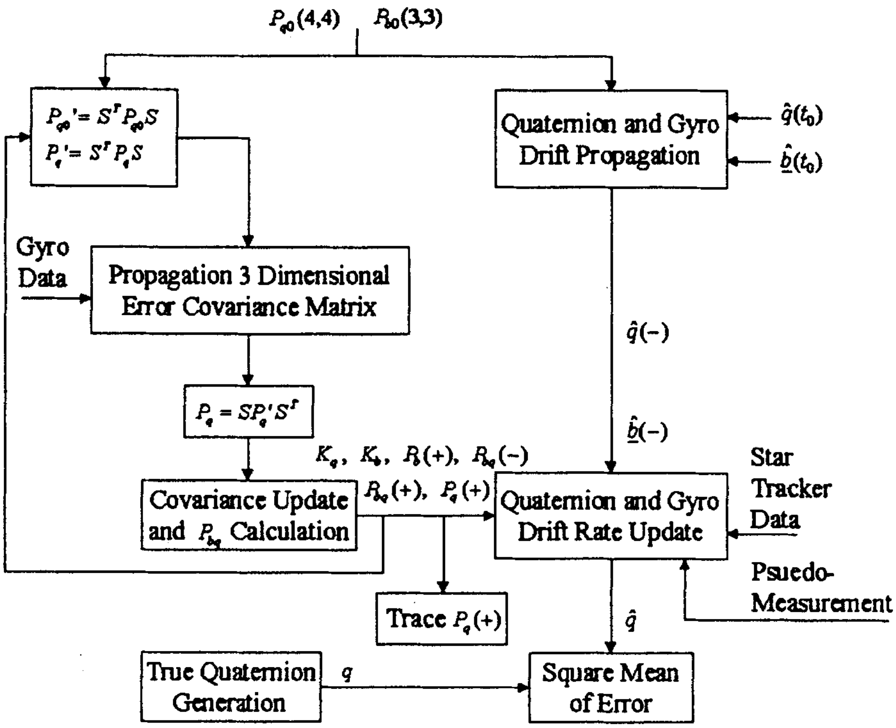

Fig. 2 Simulation flow diagram for attitude estimation.

Table 2 Initial values of the filters.

Rotation error($\delta\phi_0$)	Quaternion error($\delta \underline{q}$)	Simulation results
5°	$\delta\underline{q} = (0.025,\ 0.025,\ 0.025)^T$	Fig. 3, Fig. 4
10°	$\delta\underline{q} = (0.05,\ 0.05,\ 0.05)^T$	Fig. 5, Fig. 6
15°	$\delta\underline{q} = (0.075,\ 0.075,\ 0.075)^T$	Fig. 7, Fig. 8

standard deviation of the drift rate noise $\underline{\eta}_2$ is assumed to be 4.7×10^{-5} arc seconds/sec.

The EKF, the nonlinear filter (Yoon, *et al.*, 1999), and the proposed algorithm have been simulated under the given conditions. Since the quaternion estimation is a random process, 100 Monte Carlo simulations were carried out for the algorithms. Many simulation runs have been made and the results are shown from Figs. 3~8 according to the given initial values. The root mean square estimation errors of the quaternion against star update are plotted in figures. It can be seen from the figures that the proposed algorithm shows consistently better performance than that of EKF in all of the ranges of initial state values and covariance values of measurement. In Figs. 3~8, it is apparent that the root mean square estimation errors of the quaternion are bounded by the measurement update. Since the covariance of the filter in the proposed algorithm is compensated by the nonlinearities in the system, the root mean square estimation errors of the proposed algorithm are much lower than those of the EKF, and the convergence speed of the proposed algorithm is faster than that of the EKF. It can be seen from Figs. 3~8 that the performance of the proposed algorithm is as accurate as the nonlinear filter. However, the convergence speed of the nonlinear filter is faster than that of the proposed algorithm because of the corrupted model errors in the

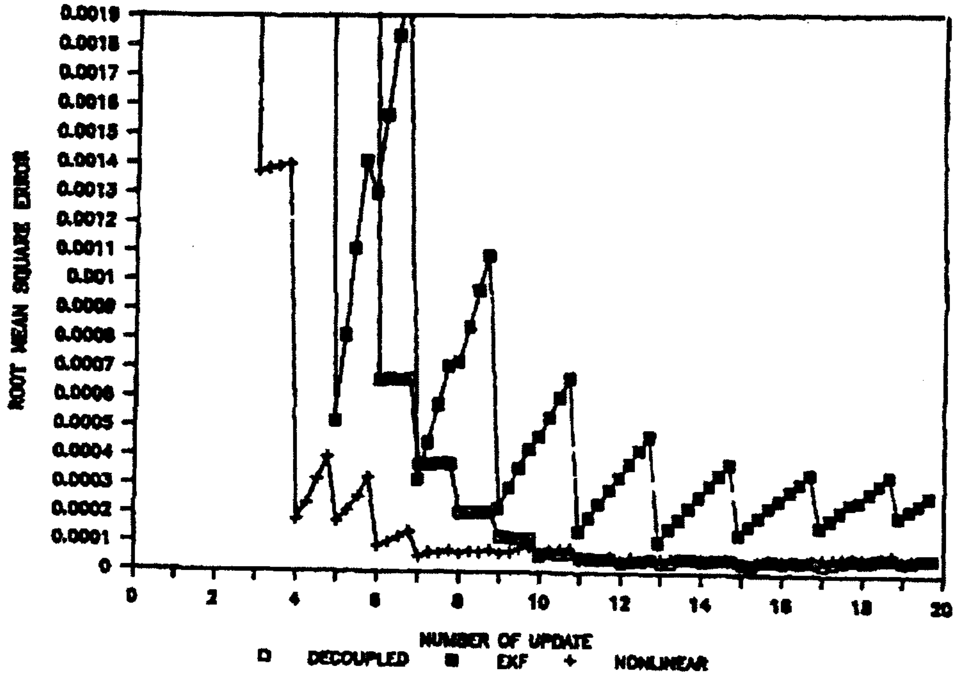

Fig. 3 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 5°, R'; 100).

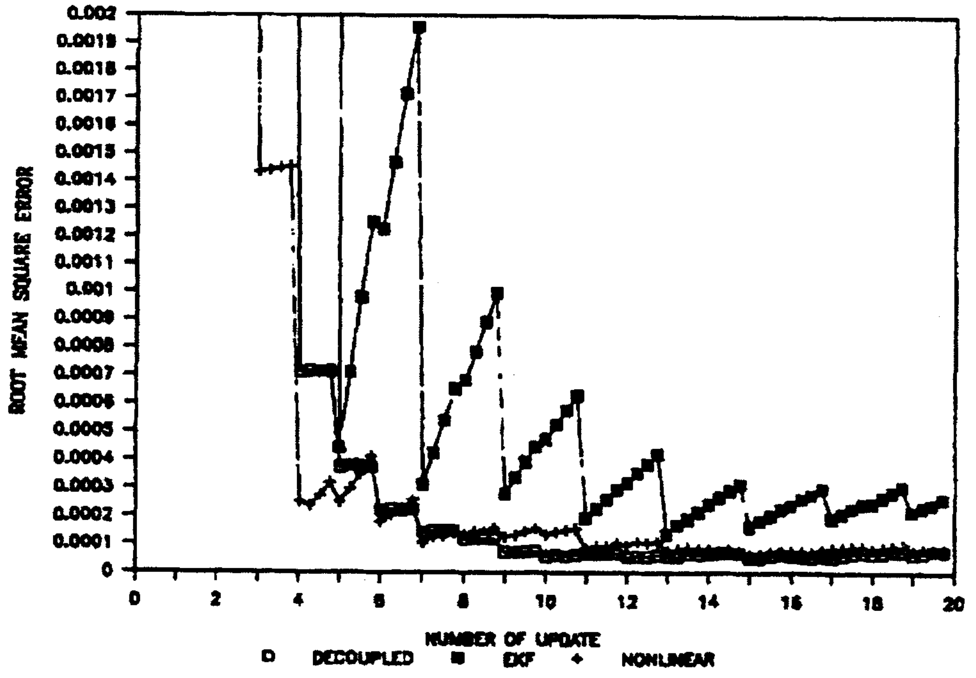

Fig. 4 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 5°, R'; 400).

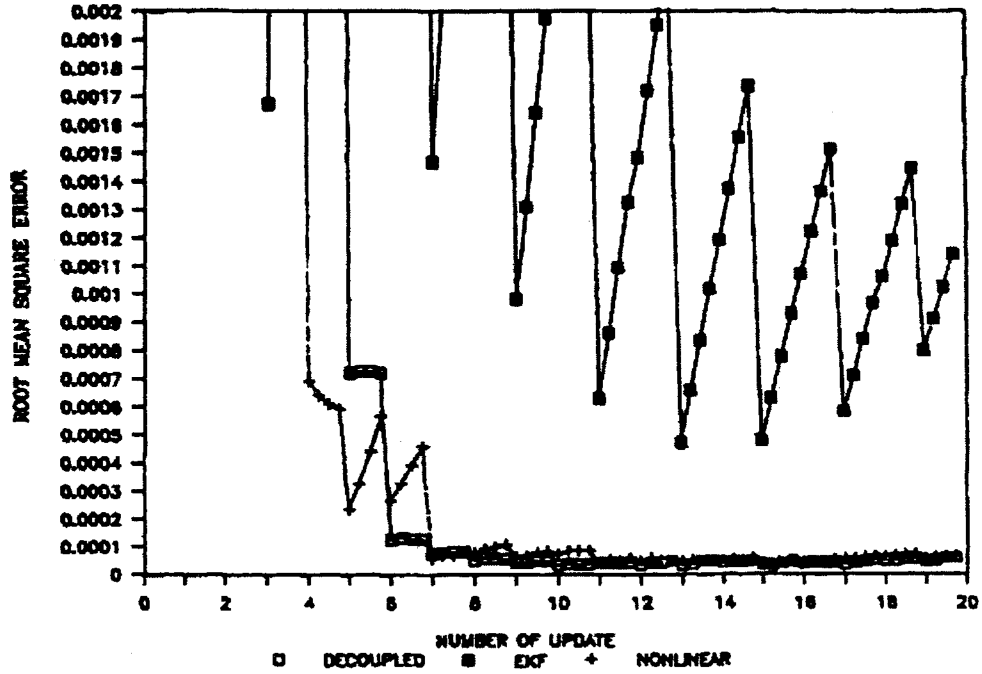

Fig. 5 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 10°, R'; 100).

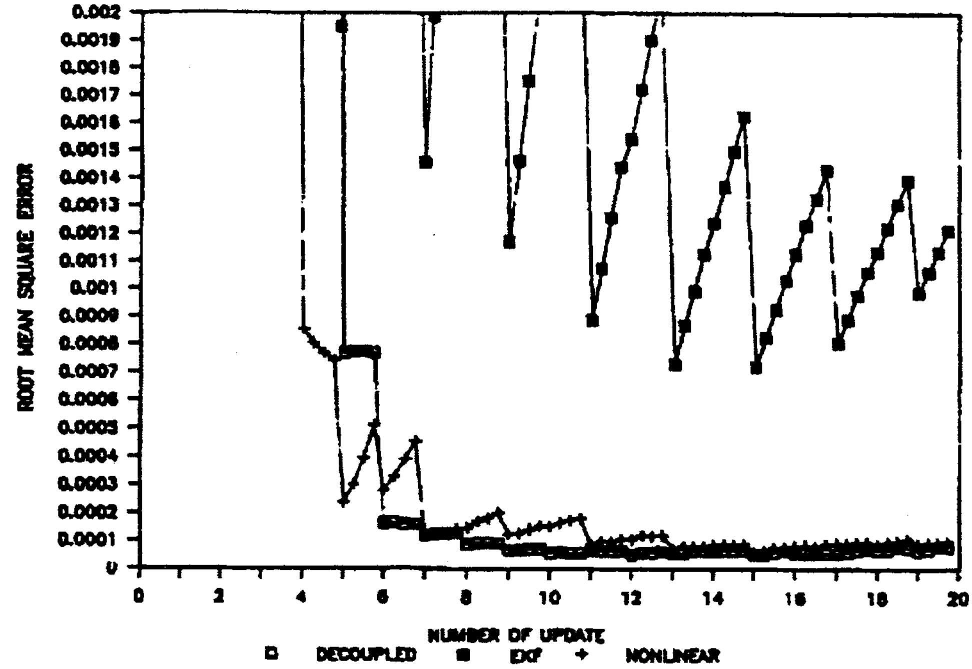

Fig. 6 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 10°, R'; 400).

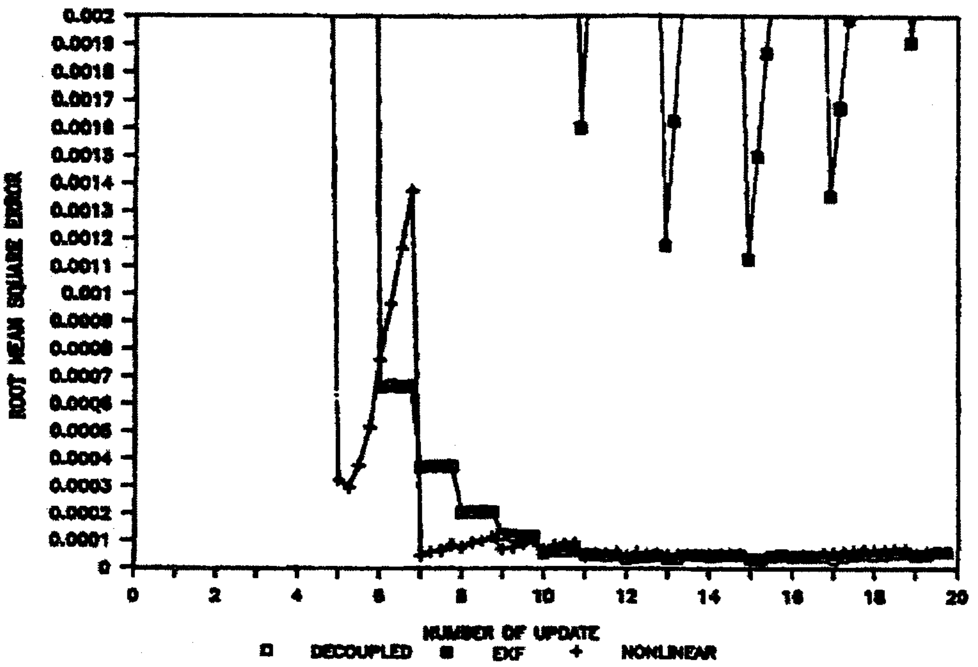

Fig. 7 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 15°, R'; 100).

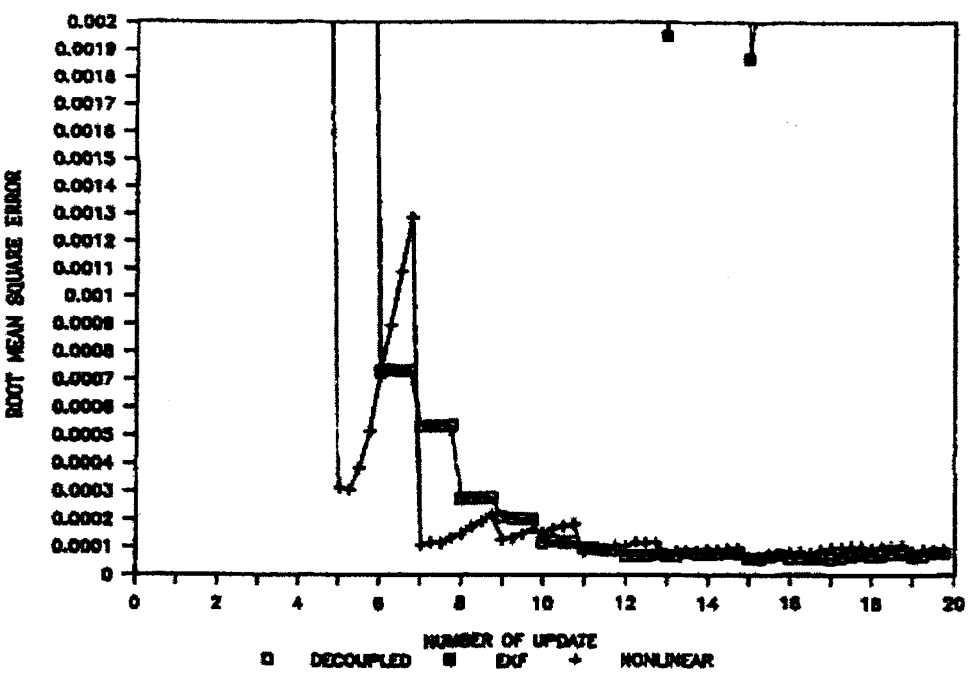

Fig. 8 The errors of the quaternion for the decoupling filter, the EKF, and the nonlinear filter ($\delta\phi_0$; 15°, R'; 400).

Table 3 Results of computation for proposed algorithms ($n_1=4$, $n_2=3$, and $r=2$).

Divison	Algorithms	Multiplications	Additions
Operation for Propagation (0.5 sec)	Nonlinear	$4n_1^2+8n_1+12n_2^3+28n_2^2+8n_2+17=713$	$4n_1n_2+11n_1+45n_2^2+15n_2=542$
	Decoupling	$4n_1^2+8n_1+8n_2^3+20n_2^2+8n_2+13=529$	$4n_1n_2+11n_1+35n_2^2+12n_2=443$
Operation for Updata (120 sec)	Nonlinear	$4(n_1+n_2)^3-6(n_1+n_2)^2+2(n_1+n_2)+5n_2^2+90n_1+n_1r+3n_2r+5n_2+3n_1^2r+4n_1r^2+2r^3+n_1n_2r+n_1^3+n_2^3+n_1^2n_2+n_2^2r+12=1907$	$4(n_1+n_2)^3-10(n_1+n_2)^2+4(n_1+n_2)+6n_2^2+91n_1-3n_1r+2n_2r+8n_2+4r+3n_1^2r+4n_1r^2+n_1n_2r+2r^2+n_1^3+2n_1^2n_2+n_2^2r-2n_1^2+3=1698$
	Decoupling	$2n_1n_2^2+2n_1^2n_2+4n_2^2+90n_1+n_1r+2n_2r+5n_2+6n_2^2+5n_2^3+2n_1^2r+2n_1r^2+2r^3+2n_2^2r+5n_2r^2+3n_1^3+17=1205$	$3n_1n_2^2+2n_1^2n_2+13n_2^2+91n_1+2n_2r-n_1r+4n_2^3+9n_2+2r+2n_1^2r+2n_1r^2+r^2+3n_1^3+n_1^3+n_2^2r+3n_2r^2+8=1198$
Total Operation for One Iteration (120 sec)	Nonlinear	173027	131769
	Decoupling	128165	107518

proposed algorithm.

In a practical application of the filter for attitude estimation, it is important to know the computation time per iteration (Bierman, 1977; Bar-Itzhack and Medan, 1983). According to the scheme presented here, the decoupling filter can be computed in pavalled since it is designed independently to have two filters form. In the simulation, the original system is 7-dimension, thus the decoupling system can be divided into 4 and 3 dimensions, and the dimension of the measurement is 2. The fourth-order Runge-Kutta method is used for numerical integration under the conditions that the propagation interval is 0.5 sec, and the update interval is 120 sec. The computation times of the nonlinear filter and the decoupling filter are summarized in Table 3 where Eq. (11a) and Eq. (13a) are applied for the propagation and Eq. (17), Eq. (18), Eq. (22), and Eq. (23) are applied for the update. n_1 and n_2 denote dimensions of the decoupled systems and r denotes the dimension of the measurement. From Table 3, we find that the computation time of the decoupling filter is improved by about 23% over the nonlinear filter.

In the problem of attitude estimation for the satellite, the attitude information generally is updated every 1 sec or 0.5 sec. Therefore, for both the nonlinear filter and the decoupling filter, real-time processing is guaranteed.

7. Concluding Remarks

The decoupling filter for attitude estimation derived in this paper is accomplished by the derived pseudo-measurement from the given measurement and the decoupled state in the original system. The proposed algorithm for real time processing, whose coupling terms are compensated through an error analysis, provides accurate attitude determination capability under a high maneuvering dynamic environment. Moreover, in a practical application of the filter, the computation time of the proposed algorithm is improved by about 23% over the nonlinear filer.

To verify the performance of the proposed algorithm with respectce to EKF and the nonlinear filter, simulations have been performed by varying the initial values of state and covariance, and the measurement covariance values. Results show that the proposed algorithm has consistently better performance than the EKF in all of the ranges of initial state values and covariance values of measurement, and is as accurate as the nonlinear filter. However, the convergence speed of the nonlinear filter is faster than the proposed algorithm because of the pseudo-measurement model errors in the proposed algorithm. Therefore, the proposed attitude estimation algorithm is useful for real time attitude estimation of a spacecraft since it has good performance and is computationally efficient.

References

Bar-Itzhack, I. Y. and Medan, Y., 1983, "Efficient Square Root Algorithm for Measurement Update in Kalman Filter," *J. Guidance, Control, and Dynamics*, Vol. 6, No. 3, pp. 129 ~ 134.

Bar-Itzhack, I. Y. and Oshman, Y., 1985, "Attitude Determination from Vector Observations: Quaternion Estimation," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. AES-21, No. 1, pp. 128 ~ 135.

Bierman, G. J., 1977, *Factorization Methods for Discrete Sequential Estimation*, Academic Press, New York.

Britting, K. R., 1971, *Inertial Navigation System Analysis*, Wiley-Interscience.

Gai, E., Daly, K., Harrison, J., and Lemos, L., 1985, "Star-Sensor-Based Satellite Attitude/Attitude Rate Estimator," *J. Guidance, Control, and Dynamics*, Vol. 8, No. 5, pp. 560 ~ 565.

Heller, W. G., 1975, "Models for Aided Inertial Navigation System Sensor Error," *TE-312-3*.

Jazwinski, A. H., 1970, *Stochastic Processes and Filtering Theory*, Academic Press, New York.

Lefferts, E. J., Markley, F. L., and Shuster, M. D., 1982, "Kalman Filtering for Spacecraft Attitude Estimation," *J. Guidance, Control, and Dynamics*, Vol. 5, pp. 417 ~ 429.

Lewis, F. L., 1986, *Optimal Estimation*, John -Wiley and Sons.

LO, J. T. -H., 1986, "Optimal Estimation for the Satellite Attitude Using Star Tracker Measurements," *Automatica*, Vol. 22, No. 4, pp. 477 ~ 482.

Miller, R. B., 1978, "Strapdown Inertial Navigation System: An Algorithm for Attitude and Navigation Computation," *AR-002-285, System Report 23*.

Nurse, R. J., Prohaska, J. T., and Riegsecker, D. G., 1978, *A New Baseline for the Inertial Navigation Strapdown Simulation*, Vol. 1, The Charles Stark Draper Laboratory, Inc., R-1136.

Sage, A. P., and Melsa, J. L., 1971, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill, New York.

Vathsal, S., 1987, "Spacecraft Attitude Determination Using a Second-Order Nonlinear Filter," *J. Guidance, Control, and Dynamics*, Vol. 10, No. 6, pp. 559 ~ 566.

Yoon, Y. J., Choi, J. W., Lee, J. G., and Fang, T. H., 1999, "An Attitude Determination Algorithm for a Spacecraft Using Nonlinear Filter," *KSME International Journal*, Vol. 13, No. 2, pp. 130 ~ 143.

Zwartbol, T., Van Den Dam, R. F., Terpstra, A. P., and Van Woerkkom, P. TH. L. M., 1985, "Attitude Estimation and Control of Maneuvering Spacecraft," *Automatica*, Vol. 21, No. 5, pp. 513 ~ 526.